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On the Potential of a Solenoidal Vector Field

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The solution to the problem of expressing a solenoidal, differentiable vector field as the curl of another vector field is given for any region deformable to star-shape external to a sphere. The given solenoidal vector field is not required to vanish at infinity. The method is then extended to a more general case of regions deformable to a star-shape region with spherical cuts, which is equivalent to that given by A. F. Stevenson (*Quart. Appl. Math.*, **12** (1954), 194–198), but without the requirement that the given solenoidal vector must vanish at infinity. The technique is then compared to another technique using Monge's potentials.

1. INTRODUCTION

The method of expressing a solenoidal, differentiable vector field $\mathbf{a}(\mathbf{x})$, whose flux over every closed surface vanishes, as the curl of another vector field $\mathbf{b}(\mathbf{x})$, i.e.,

$$\nabla \times \mathbf{b} = \mathbf{a}(\mathbf{x}), \tag{1.1}$$

is a central device in the solutions of many problems in different branches of mathematical physics such as electromagnetism, elasticity, and fluid mechanics. Such an expression is an old problem, which appeared to have been exhaustively investigated by many authors. However, this author has come across another type of solution to the problem, which may be regarded as an extension of the classical integration method for differential forms in star-shaped regions to cover regions deformable to a star-shaped region with spherical cuts, each of which is external to every other and is wholly within the star-shaped region. The solution here does not require that $\mathbf{a}(\mathbf{x})$ vanishes at infinity as do those in current literature.

The well-known solutions to the problem can be found in classical textbooks by Goursat [1], Picard [2], Weatherburn [3], Courant [4], and Lass [5]. However, it is pointed out by Stevenson [6] that the region under consideration in these texts is either the whole space or is not men-

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tioned at all (Lichtenstein's book [7] was not included in the survey by Stevenson, and is an exception to Stevenson's comment as it is similar to Stevenson's work). Stevenson hence provided a method which is applicable to any simply connected, finite region between a smooth external boundary surface and a number of smooth internal boundary surfaces, each of the later being external to every other. His method is based on the solutions of Neumann boundary value problems for the boundary surfaces. Stevenson then claimed that the method can be extended to an infinite region even if the given vector function $\mathbf{a}(\mathbf{x})$ does not vanish at infinity. Despite claims to the contrary by its author, the method is inconclusive when applied to an infinite region if the vector function $\mathbf{a}(\mathbf{x})$ does not vanish at infinity to the order $O(|\mathbf{x}|^{-2-\varepsilon})$, where ε is a positive number. This inconclusiveness is caused by the lack of a proof that the sequence of vector functions $\{\mathbf{b}(\mathbf{x})\}$ constructed by the method converges pointwise to a vector function $\mathbf{b}_{\infty}(\mathbf{x})$ which is independent of the external boundary of the region as this boundary grows larger (The limitation is not peculiar only to Stevenson's method but is inherent in any method which is based on a volume integral of some function to satisfy a Poisson equation involving that same function; Lichtenstein [7] had pointed out this limitation in his work). Although Stevenson's proof is less restrictive than simple proofs, such as those found in Courant's text [4], the requirement that $\mathbf{a}(\mathbf{x})$ must vanish at infinity to the order of at least $O(|\mathbf{x}|^{-2-\varepsilon})$ does exclude its application in some practical problems where such a requirement on the property of $\mathbf{a}(\mathbf{x})$ cannot be satisfied.

Quite another line of proof of this theorem, which is apparently unrelated to those mentioned above, is known as the Converse of Poincaré's Lemma to differential forms, and can be found in textbooks; for example, see Flanders's book [8]. This line of proof originates from the study of differential forms, which was first investigated in its general form by Pfaff [9] and later perfected by Mayer [10], Morera [11], Severi [12], etc. In this method, the only requirement for the application of the theorem is that the region under consideration must be *star-shaped*, centered on a point, or *deformable to star shape*, with the application of a more general system of coordinates (these terms are defined in Section 2). The method does not require that the vector $\mathbf{a}(\mathbf{x})$ vanish at infinity as in Stevenson's method, but it is applicable only to star-shaped regions and is therefore excluded from periphractic regions. (A region is defined to be *periphractic* if it has internal boundaries. A region is *non-periphractic* if every closed surface in it can be continuously shrunk to a single point without leaving the region.) This line of proof relies heavily on formal devices and hence the physical meaning is obscured and the proof does not allow easy extension to regions other than those deformable to star-shape.

In this paper, a new approach to the problem is presented. This approach gives a unifying view of the established ones. The method will be essentially the simple method given in Courant's text, subjected to a few tensorial transformations. The result turns out to be an extension of the method of Mayer, and it reduces to the well-known formula used in the proof of the Converse of Poincaré's Lemma when the region under consideration is star-shaped. The proof is applicable to a region *deformable to star-shape external to a sphere* or to a region *deformable* to a star-shaped one with spherical cuts (these terms are defined in Section 2), and it does not require the given vector field $\mathbf{a}(\mathbf{x})$ to vanish at infinity as does Stevenson's proof.

The result is then compared to the representation of an arbitrarily given solenoidal, differentiable vector field as the vector product of the gradients of two potential functions, which are called the Monge potentials. This method has been used in an early work be Clebsch [13, 14] now known as Clebsch' transformation (see Lamb's book [15]). The limitation of such a decomposition, which is a crucial point in Clebsch' transformation, is then pointed out.

Finally, the result is used to derive a less restrictive version of the Stokes–Helmholtz decomposition theorem for a simply connected, infinite region deformable to star-shape with spherical cuts than the conventional version. Some interesting examples of the application to practical problems of the line integration technique used here are also mentioned.

2. Solution in a Region Deformable to Star-Shape External to a Sphere

2.1. *Mathematical Preliminaries*

As the nature of this problem is basically a tensorial one, the method employed in this paper will be that of tensor calculus. The common notations of tensor calculus are advantageous here as they allow one to change from one coordinate system to another without having to redefine various quantities and they give more comprehensible formulations. The convention used in Lass' book [5] is adopted here and is summarised as: A bold roman letter denotes a tensor or a vector and its corresponding non-bold letter with attached indices denotes its components in a coordinate system. The cartesian coordinates of the Euclidean space are denoted by (x^1, x^2, x^3). A bar on top of a scalar quantity denotes that same quantity as a function of the coordinate system $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$, e.g.,

$$\bar{\phi}(\bar{x}^1, \bar{x}^2, \bar{x}^3) \equiv \phi(x^1(\bar{x}^1, \bar{x}^2, \bar{x}^3), x^2(\bar{x}^1, \bar{x}^2, \bar{x}^3), x^3(\bar{x}^1, \bar{x}^2, \bar{x}^3)).$$

A bar on top of a tensorial component denotes a particular component, in the coordinate system $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$, of that same tensor as a function of the coordinate system $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$. Summation convention is used so that repeated indices in the same term denote a sum over all allowed values of indices. Superscripts and subscripts are used for contravariant and covariant variables, respectively. For example, x^i is the component of the contravariant vector $\mathbf{x}$.

Let $\mathbf{T}$ be an absolute tensor with components $T^{\alpha_1\alpha_2\ldots\alpha_r}_{\beta_1\beta_2\ldots\beta_s}$; then its covariant derivative with respect to the contravariant coordinate x^m is given by

$$T^{\alpha_1\alpha_2\ldots\alpha_r}_{\beta_1\beta_2\ldots\beta_s,m} \equiv \frac{\partial T^{\alpha_1\alpha_2\ldots\alpha_r}_{\beta_1\beta_2\ldots\beta_s}}{\partial x^m} + T^{\mu\alpha_2\ldots\alpha_r}_{\beta_1\beta_2\ldots\beta_s}\Gamma^{\alpha_1}_{\mu m} + \cdots + T^{\alpha_1\alpha_2\ldots\mu}_{\beta_1\beta_2\ldots\beta_s}\Gamma^{\alpha_r}_{\mu m}$$
$$- T^{\alpha_1\alpha_2\ldots\alpha_r}_{\mu\beta_2\ldots\beta_s}\Gamma^{\mu}_{\beta_1 m} - \cdots - T^{\alpha_1\alpha_2\ldots\alpha_r}_{\beta_1\beta_2\ldots\mu}\Gamma^{\mu}_{\beta_s m}, \tag{2.1}$$

which is an absolute tensor of covariant order one greater than $T^{\alpha_1\alpha_2\ldots\alpha_r}_{\beta_1\beta_2\ldots\beta_s}$.

Due to the symmetry of the Christoffel symbols $\Gamma^{\gamma}_{\alpha\beta}$ with respect to their two lower indices, we have the curl of an absolute covariant vector $\mathbf{b}$ as

$$(\text{curl } \mathbf{b})_{ij} \equiv b_{i,j} - b_{j,i} = \frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}, \tag{2.2}$$

which is a twice covariant antisymmetric tensor.

Similarly, the symmetry of the Christoffel symbols $\Gamma^{\gamma}_{\alpha\beta}$ with respect to their two lower indices also gives the relation

$$c_{12,3} + c_{23,1} + c_{31,2} = \frac{\partial c_{12}}{\partial x^3} + \frac{\partial c_{23}}{\partial x^1} + \frac{\partial c_{31}}{\partial x^2}.$$

This allows a definition of the "exterior derivative" of a twice covariant antisymmetric tensor so that the exterior derivative is free from the Christoffel symbols. Let c_{ij} be the components of a twice covariant antisymmetric tensor $\mathbf{c}(\mathbf{x})$. Its exterior derivative $d\mathbf{c}$ is a thrice covariant, antisymmetric tensor defined by

$$d\mathbf{c} \equiv \frac{1}{2}\frac{\partial c_{ij}}{\partial x^k}\mathbf{e}^k \wedge \mathbf{e}^i \wedge \mathbf{e}^j,$$

where $(\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3)$ denotes the natural contravariant basis of the coordinate system (x^1, x^2, x^3) and $\wedge$ denotes the exterior product (Bowen and Wang [16, p. 303]). The above expression can be rewritten as

$$d\mathbf{c} \equiv \frac{1}{2}\delta^{123}_{ijk}\frac{\partial c_{ij}}{\partial x^k}\mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3, \tag{2.3}$$

where δ^{123}_{ijk} is a permutation symbol having the value of $+1$, -1, 0

depending on whether (ijk) is an even, an odd, or not a permutation of (123). It then gives

$$d\mathbf{c} \equiv \left(\frac{\partial c_{12}}{\partial x^3} + \frac{\partial c_{23}}{\partial x^1} + \frac{\partial c_{31}}{\partial x^2}\right) \mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3 = (c_{12,3} + c_{23,1} + c_{31,2})\, \mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3,$$

which shows that $d\mathbf{c}$ is a thrice covariant, antisymmetric tensor with only one single component.

For any given twice covariant antisymmetric tensor with components c_{jk}, we can always form a contravariant vector with components a^i, using the formula

$$a^i = \frac{1}{(\det(g))^{1/2}}\, \delta^{123}_{ijk}\, c_{jk}, \tag{2.4a}$$

with g_{ij} being the components of the twice covariant metric tensor $\mathbf{g}$ for the coordinate system. Conversely, for every given contravariant vector a^i, we can always form a twice covariant antisymmetric tensor c_{jk}, using

$$c_{jk} = \tfrac{1}{2}(\det(g))^{1/2}\, \delta^{123}_{ijk}\, a^i. \tag{2.4b}$$

The tensors $\mathbf{a}$ and $\mathbf{c}$ so related are called the dual of each other in three-dimensional coordinates (It is worth mentioning that the tensorial problem considered here can also be rewritten wholly in vectorial notations at the risk of losing the insight into the methodology. The reader who wants to follow such an approach can use Eq. (2.4b) to replace terms such as $\bar{c}_{12}(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ by

$$\frac{1}{2}\frac{\partial(x^1, x^2, x^3)}{\partial(\bar{x}^1, \bar{x}^2, \bar{x}^3)}\, \bar{a}^3(\bar{x}^1, \bar{x}^2, \bar{x}^3),$$

where $\partial(x^1, x^2, x^3)/\partial(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ denotes the Jacobian of the coordinate transformation, every time it occurs).

We next have a number of definitions.

DEFINITION 1. A *domain* is an open set, any two of whose points can be joined by a polygonal line, of a finite number of segments, all of whose points belong to the set (Kellogg [17, p. 93]).

DEFINITION 2. A *region* is either a domain or a domain together with some or all of its boundary points (Kellogg [17, p. 93]).

DEFINITION 3. A function $f(\mathbf{x})$ is of class $C^n(D)$ if it is defined, continuous together with all of its partial derivatives of order up to and including n $(n \geqslant 0)$ in the region D.

We write $f(\mathbf{x}) \in C^n(D)$ to denote that $f(\mathbf{x})$ is of class $C^n(D)$.

DEFINITION 4. A coordinate system $(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)$ is non-singular in a region E with respect to a coordinate system (x^1, x^2, x^3) if the Jacobian

$$\frac{\partial(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)}{\partial(x^1, x^2, x^3)}$$

is neither infinite nor zero in the region E.

DEFINITION 5. A region E is *deformable* to a particular geometry with respect to the coordinates (x^1, x^2, x^3) if we can draw a coordinate system $(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)$ which is of class $C^2(E)$ and is non-singular in E with respect to (x^1, x^2, x^3), such that when E is drawn with $(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)$ serving as the cartesian coordinate system, E assumes that geometry.

The operation of drawing E with $(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)$ serving as the cartesian coordinates is called a *deformation* of the old geometry into the new geometry.

For example, define $r = ((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}$ and $R = ((\bar{\bar{x}}^1)^2 + (\bar{\bar{x}}^2)^2 + (\bar{\bar{x}}^3)^2)^{1/2}$ and let

$$\frac{\bar{\bar{x}}^i - \bar{\bar{a}}^i}{((\bar{\bar{x}}^1 - \bar{\bar{a}}^1)^2 + (\bar{\bar{x}}^2 - \bar{\bar{a}}^2)^2 + (\bar{\bar{x}}^3 - \bar{\bar{a}}^3)^2)^{1/2}} = \frac{x^i - b^i}{((x^1 - b^1)^2 + (x^2 - b^2)^2 + (x^3 - b^3)^2)^{1/2}}$$

and

$$R = R(r),$$

where $R(r)$ is a strictly monotonic, twice differentiable function defined on the interval (r_0, r_1); then we have the spherical shell between the two radii r_0, r_1 deformable to the spherical shell between the two radii $R(r_0)$ and $R(r_1)$.

If we choose $R(r)$ to be

$$R = mr \quad \text{for} \quad r \leqslant r_3, \qquad R = r \quad \text{for} \quad r \geqslant r_2,$$

$$R = mr + \frac{(1-m)\,r}{\int_{r_3}^{r_2} \exp[-1/(r_2 - t)(t - r_3)]\,dt} \times \int_{r_3}^{r} \exp\left[\frac{-1}{(r_2 - t)(t - r_3)}\right] dt \qquad \text{for} \quad r_3 < r < r_2,$$

where m is a positive constant less than unity, $0 < m < 1$, and r_2, r_3 are two positive constants with $0 < r_3 < r_2$, then we have the regions exterior and interior to the spherical surface of radius r_4 deformable respectively to the regions exterior and interior to the spherical surface of radius $R(r_4) < r_4$, where r_4 is an arbitrary positive constant less than r_2, $0 < r_4 < r_2$. This deformation transformation does not alter the geometry of any region exterior to the spherical surface of radius r_2. It is worth noting that the above relationship between r and R is based on a function $f(t)$ defined by

$$f(t) = \exp\left(\frac{-1}{1-t^2}\right) \quad \text{if} \quad |t| < 1 \qquad \text{and} \qquad f(t) = 0 \quad \text{if} \quad |t| \geqslant 1,$$

which is differentiable to any order for all t. This function and its indefinite integral $\int_{-\infty}^{t} f(s)\, ds$ are quite useful in forming from a given system of coordinates (x^1, x^2, x^3) a new system $(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)$ which is non-singular and differentiable to any order with respect to the given system of coordinates (x^1, x^2, x^3).

Definition 6. A region D in the cartesian coordinate system (x^1, x^2, x^3) is called *star-shaped* centered on the point W and with respect to the coordinate system (x^1, x^2, x^3) if W is not on the boundary ∂D of D and if every point of D can be joined to the point W by a line segment which lies wholly in D. When the point W is not mentioned, the star-shapeness is understood to be centered on the origin O. The region D can be bounded or unbounded.

The above definition requires that W must be an interior point of the region D.

Definition 7. A region E is called *deformable to star-shaped* centered on a point W, and with respect to the coordinates (x^1, x^2, x^3) if we can draw a coordinate system $(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)$, which is of class $C^2(E)$ and non-singular in E with respect to (x^1, x^2, x^3), such that when E is drawn with $(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)$ serving as the cartesian coordinate system, it is star-shaped centered on the point W and with respect to this new coordinate system $(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)$. When W is not mentioned, the star-shapeness is understood to be centered on the origin O.

Definition 8. A region D is called *star-shaped external to a sphere* S with respect to the coordinate system (x^1, x^2, x^3) if it is the difference $G - H$ between a region G and a region H. G is star-shaped centered on W and, with respect to the coordinate system (x^1, x^2, x^3), W is the center of the sphere S which lies wholly in G, and H is the interior of the sphere S. Such a region D is illustrated in Fig. 1.

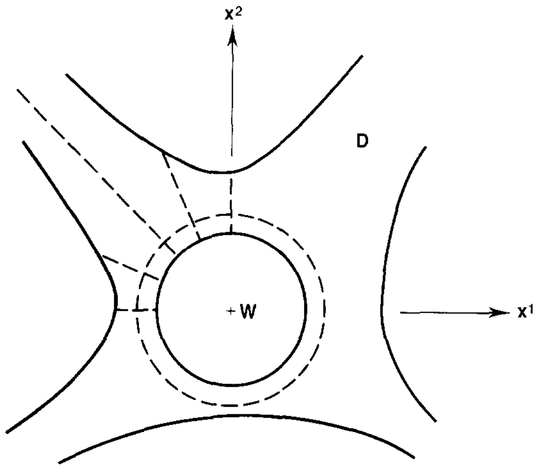

FIG. 1. A region D star-shaped external to a sphere S.

DEFINITION 9. A region E is *deformable to star-shape external to a sphere* S with respect to the coordinate system (x^1, x^2, x^3) if we can draw a coordinate system $(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)$, which is of class $C^2(E)$ and non-singular in E with respect to (x^1, x^2, x^3), such that, when E is drawn with $(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)$ serving as the cartesian coordinate system, it is star-shaped external to the sphere S with respect to this new coordinate system $(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)$. Such a region E is illustrated in Fig. 2.

DEFINITION 10. A vector function $\mathbf{a}(\mathbf{x})$ is divergence-free in a region D if $\partial a^1/\partial x^1 + \partial a^2/\partial x^2 + \partial a^3/\partial x^3 = 0$ at every point of D.

DEFINITION 11. A vector function $\mathbf{a}(\mathbf{x})$ is solenoidal in a region D if $\int_S \mathbf{a}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, dS(\mathbf{x}) = 0$ for every closed surface S in D, where $\mathbf{n}(\mathbf{x})$ is the normal vector of the surface S.

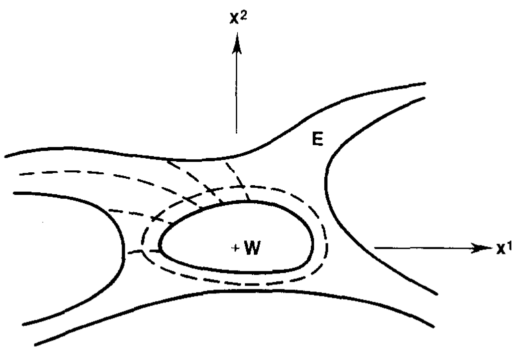

FIG. 2. A region E deformable to star-shape external to a sphere.

We note that every solenoidal, differential vector function in a region D is divergence-free in D, but the converse is not true if D has an internal surface. In this latter case, every divergence-free vector function in D is also solenoidal in D if its flux over the internal surface is zero.

DEFINITION 12. Let S be a parametric two-dimensional surface defined by $S: (\phi_1, \phi_2) \in Q \subset R^2 \to (x^1, x^2, x^3)$. The surface integral of a twice covariant tensor $\mathbf{c}$ on S (the flux Φ of $\mathbf{c}$ on S) is defined to be

$$\Phi \equiv \int_S \bar{\bar{c}}_{ij}\, d\bar{\bar{x}}^i\, d\bar{\bar{x}}^j \equiv \int_Q \bar{\bar{c}}_{ij} \frac{\partial(\bar{\bar{x}}^i \bar{\bar{x}}^j)}{\partial(\phi_1, \phi_2)}\, d\phi_1\, d\phi_2. \tag{2.5}$$

We apply the Binet–Cauchy product formula to the above equation to obtain

$$\Phi = \frac{1}{2} \int_Q \bar{\bar{c}}_{ij} \frac{\partial(\bar{\bar{x}}^i \bar{\bar{x}}^j)}{\partial(x^b, x^c)} \frac{\partial(x^b, x^c)}{\partial(\phi_1, \phi_2)}\, d\phi_1\, d\phi_2.$$

Hence

$$\begin{aligned}
\Phi &= \frac{1}{2 \times 6} \int_Q \bar{\bar{c}}_{ij}\, \delta^{ijk}_{abc}\, \delta^{abc}_{ijk} \frac{\partial(\bar{\bar{x}}^i \bar{\bar{x}}^j)}{\partial(x^b, x^c)} \frac{\partial(x^b, x^c)}{\partial(\phi_1, \phi_2)}\, d\phi_1\, d\phi_2 \\
&= \frac{1}{2 \times 6} \int_Q \bar{\bar{c}}_{ij}\, \delta^{ijk}_{abc} \frac{\partial x^a}{\partial \bar{\bar{x}}^k} \frac{\partial(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)}{\partial(x^1, x^2, x^3)} \frac{\partial(x^b, x^c)}{\partial(\phi_1, \phi_2)}\, d\phi_1\, d\phi_2 \\
&= \frac{1}{2 \times 6} \int_Q c_{lm} \left(\delta^{ijk}_{abc} \frac{\partial x^l}{\partial \bar{\bar{x}}^i} \frac{\partial x^m}{\partial \bar{\bar{x}}^j} \frac{\partial x^a}{\partial \bar{\bar{x}}^k} \right) \\
&\quad \times \frac{\partial(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)}{\partial(x^1, x^2, x^3)} \frac{\partial(x^b, x^c)}{\partial(\phi_1, \phi_2)}\, d\phi_1\, d\phi_2 \\
&= \frac{1}{2} \int_Q c_{lm}\, \delta^{lm}_{bc} \frac{\partial(x^b, x^c)}{\partial(\phi_1, \phi_2)}\, d\phi_1\, d\phi_2 = \int_Q c_{lm} \frac{\partial(x^l, x^m)}{\partial(\phi_1, \phi_2)}\, d\phi_1\, d\phi_2.
\end{aligned}$$

Therefore the flux Φ is defined independent of the coordinates in use, and (2.5) is a tensorial equation.

If we replace $\mathbf{c}$ in Eq. (2.5) by its dual vector $\mathbf{a}$, we obtain

$$\begin{aligned}
\Phi &= \frac{1}{2} \int_S a^k\, \delta^{123}_{klm} \frac{\partial(x^l, x^m)}{\partial(\phi_1, \phi_2)}\, d\phi_1\, d\phi_2 \\
&= \frac{1}{2} \int_S \frac{\partial(x^1, x^2, x^3)}{\partial(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)}\, \bar{\bar{a}}^k\, \delta^{123}_{klm} \frac{\partial(\bar{\bar{x}}^l, \bar{\bar{x}}^m)}{\partial(\phi_1, \phi_2)}\, d\phi_1\, d\phi_2,
\end{aligned} \tag{2.6}$$

which means that the flux of $\mathbf{a}$ with (x^1, x^2, x^3) serving as cartesian coordinates is equal to the flux of $\mathbf{a}[\partial(x^1, x^2, x^3)/\partial(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)]$ with $(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)$ serving as cartesian coordinates. Therefore, the following definition which is independent of the coordinate system is preferred.

DEFINITION 13. A twice covariant tensor $\mathbf{c}(\mathbf{x})$ is of *zero-period* in a region D if $\int_S c_{ij}(\mathbf{x})\, dx^i\, dx^j = 0$ for every closed surface S in D. (This terminology is adapted from the term "period" used in Flanders' book [8].)

We are now ready to embark on the problem. We will first solve the problem for the simple case of a region D which is star-shape external to a sphere. The methodology so developed will then be applied to the more complicated case of a region E deformable to star-shape external to a sphere.

2.2. *A Star-Shaped Region External to a Sphere*

We now prove the existence of the vector $\mathbf{b}(\mathbf{x})$ satisfying Eq. (1.1) for any given solenoidal, differential vector function $\mathbf{a}(\mathbf{x})$ in a region D star-shaped external to a sphere with respect to the coordinate system (x^1, x^2, x^3).

Let $\mathbf{a}(\mathbf{x})$ be an arbitrarily given solenoidal, differentiable vector field in cartesian coordinates. We then have this vector field divergence-free, i.e.,

$$\frac{\partial a^1}{\partial x^1} + \frac{\partial a^2}{\partial x^2} + \frac{\partial a^3}{\partial x^3} = 0. \tag{2.7}$$

Form the twice covariant antisymmetric tensor $\mathbf{c}(\mathbf{x})$, which is the dual to $\mathbf{a}(\mathbf{x})$. We also have its exterior derivative vanishing in cartesian coordinates, i.e.,

$$c_{12,3} + c_{23,1} + c_{31,2} = 0.$$

Since this equation is a tensorial one as is Eq. (2.3), it holds in all coordinate systems. Thus

$$\bar{c}_{12,3} + \bar{c}_{23,1} + \bar{c}_{31,2} = 0, \tag{2.8}$$

which facilitates the solution quite a lot as we can choose any coordinate system convenient to the task.

We now find the solution to the tensorial equation

$$\frac{\partial \bar{b}_i}{\partial \bar{x}^j} - \frac{\partial \bar{b}_j}{\partial \bar{x}^i} = \bar{c}_{ij} \tag{2.9}$$

given the condition (2.8). We can choose $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ such that $\bar{x}^3$ is along the radial direction of the (x^1, x^2, x^3) coordinates and $\bar{x}^1, \bar{x}^2$ are the longitude and latitude angles of the (x^1, x^2, x^3) coordinates. In the new coordinates $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$, the region under consideration can be joined by a line along the direction $\bar{x}^3$ to a surface of $\bar{x}^3 = r_0$. Therefore we use the method as given in Courant's book to construct the covariant vector $\mathbf{b}$ as in the following.

We are only looking for a particular solution to, and not all solutions of, Eq. (2.9). Therefore, we are at liberty to use any convenient method to obtain a solution. Here, we choose $\bar{b}_3 = 0$, to obtain

$$\bar{b}_1 = \int_{r_0}^{\bar{x}^3} \bar{c}_{31}(\bar{x}^1, \bar{x}^2, \zeta)\, d\zeta + \bar{\lambda}_1(\bar{x}^1, \bar{x}^2), \tag{2.10a}$$

$$\bar{b}_2 = -\int_{r_0}^{\bar{x}^3} \bar{c}_{23}(\bar{x}^1, \bar{x}^2, \zeta)\, d\zeta + \bar{\lambda}_2(\bar{x}^1, \bar{x}^2), \tag{2.10b}$$

$$\bar{b}_3 = 0, \tag{2.10c}$$

and solve for the solution to (2.9) by letting $\bar{\lambda}_1(\bar{x}^1, \bar{x}^2)$ and $\bar{\lambda}_2(\bar{x}^1, \bar{x}^2)$ satisfy the equation

$$\frac{\partial \bar{\lambda}_2(\bar{x}^1, \bar{x}^2)}{\partial \bar{x}^1} - \frac{\partial \bar{\lambda}_1(\bar{x}^1, \bar{x}^2)}{\partial \bar{x}^2} = \bar{c}_{12}(\bar{x}^1, \bar{x}^2, r_0). \tag{2.11}$$

The problem has thus been simplified to the solution of a two-dimensional vector function of two variables with its curl given. The existence of the solution to this equation on a spherical surface is postponed until after the derivation of the following equation (2.13).

Application of a tensorial transformation to the components $\bar{b}_i$ of the solution vector $\mathbf{b}$ so found gives

$$\bar{b}_1 = \frac{\partial x^i}{\partial \bar{x}^1} \frac{x^j}{r} \int_{r_0/r}^{1} r t c_{ij}(t\mathbf{x})\, dt + \bar{\lambda}_1(\bar{x}^1, \bar{x}^2) \tag{2.12a}$$

$$\bar{b}_2 = -\frac{\partial x^i}{\partial \bar{x}^2} \frac{x^j}{r} \int_{r_0/r}^{1} r t c_{ij}(t\mathbf{x})\, dt + \bar{\lambda}_2(\bar{x}^1, \bar{x}^2), \tag{2.12b}$$

where r denotes $[(x^1)^2 + (x^2)^2 + (x^3)^2]^{1/2}$ and r_0 is the radius of the sphere S centered at the origin of the (x^1, x^2, x^3) coordinates. The contravariant vector $\mathbf{b}(\mathbf{x})$ is given in the final form as

$$\begin{aligned} b_1 = x^3 \int_{r_0/r}^{1} t c_{31}(t\mathbf{x})\, dt - x^2 \int_{r_0/r}^{1} t c_{12}(t\mathbf{x})\, dt \\ + \frac{\partial \bar{x}^1}{\partial x^1} \bar{\lambda}_1(\bar{x}^1, \bar{x}^2) + \frac{\partial \bar{x}^2}{\partial x^1} \bar{\lambda}_2(\bar{x}^1, \bar{x}^2), \end{aligned} \tag{2.13a}$$

$$b_2 = x^1 \int_{r_0/r}^{1} tc_{12}(t\mathbf{x})\,dt - x^3 \int_{r_0/r}^{1} tc_{23}(t\mathbf{x})\,dt + \frac{\partial \bar{x}^1}{\partial x^2}\bar{\lambda}_1(\bar{x}^1, \bar{x}^2) + \frac{\partial \bar{x}^2}{\partial x^2}\bar{\lambda}_2(\bar{x}^1, \bar{x}^2), \tag{2.13b}$$

$$b_3 = x^2 \int_{r_0/r}^{1} tc_{23}(t\mathbf{x})\,dt - x^1 \int_{r_0/r}^{1} tc_{31}(t\mathbf{x})\,dt + \frac{\partial \bar{x}^1}{\partial x^3}\bar{\lambda}_1(\bar{x}^1, \bar{x}^2) + \frac{\partial \bar{x}^2}{\partial x^3}\bar{\lambda}_2(\bar{x}^1, \bar{x}^2), \tag{2.13c}$$

which is the required solution to the problem in a star-shaped region external to a sphere if we can satisfy Eq. (2.11) on the surface of this sphere. For the special case where $r_0 = 0$ and $\bar{\lambda}_1$, $\bar{\lambda}_2$ are both identically zero, Eq. (2.13) becomes the familiar integration formula, which is attributed to Mayer, used in the proof of the Converse of Poincaré's Lemma for a star-shaped region (for example, see Flanders' book [8]); this proof for the basic case of star-shaped regions does not rely on any relation of the form (2.11).

We now proceed to prove that there exist the functions $\bar{\lambda}_1$ and $\bar{\lambda}_2$ satisfying Eq. (2.11) if the integral $\int_S \mathbf{a}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, dS(\mathbf{x})$ over the surface S of the vector $\mathbf{a}(\mathbf{x})$ dual to $\mathbf{c}$ vanishes with $\mathbf{n}$ being the normal unit vector of the spherical surface S. For this given vector $\mathbf{a}$ on the surface of the sphere, the solution to the internal Neumann problem gives a harmonic function $\phi(\mathbf{x})$, which has its normal component on the surface S equal to the normal component of $\mathbf{a}$. Consider now the vector function

$$\mathbf{v}(\mathbf{x}) = \nabla\phi,$$

defined on the interior of S. This function is differentiable and divergence-free in a star-shaped region, therefore it is equal to the curl of another vector $\mathbf{w}(\mathbf{x})$. This is the result of the theorem for the basic case of a star-shaped region. (Alternatively, we can use Lichtenstein's or Stevenson's method to determine the vector $\mathbf{w}(\mathbf{x})$ in this finite region bounded by S.) The components of $\nabla \times \mathbf{w}$ are equal to $\mathbf{v}$ and therefore satisfy

$$\frac{\partial w_i}{\partial x^j} - \frac{\partial w_j}{\partial x^i} = (\det(\mathbf{g}))^{1/2}\, v^k\, \delta_{ijk}^{123},$$

in cartesian coordinates, inside the sphere and on its surface. Hence there is a solution to (2.11) on the surface of the sphere, which is

$$\bar{\lambda}_1(\bar{x}^1, \bar{x}^2) = \frac{1}{2}\frac{\partial x^i}{\partial \bar{x}^1}\, w_i(x^1 r_0/r, x^2 r_0/r, x^3 r_0/r) \tag{2.14a}$$

and

$$\bar{\lambda}_2(\bar{x}^1, \bar{x}^2) = \frac{1}{2}\frac{\partial x^i}{\partial \bar{x}^2}\, w_i(x^1 r_0/r, x^2 r_0/r, x^3 r_0/r). \tag{2.14b}$$

Thus for any solenoidal, differentiable vector $\mathbf{a}(\mathbf{x})$ in a star-shaped region external to a sphere, there is a vector function $\mathbf{b}(\mathbf{x})$, the curl of which is equal to $\mathbf{a}(\mathbf{x})$.

The derivation from Eqs. (2.12) to (2.13) of this subsection can also be carried out using spherical coordinates rather than the general tensor machinery. However, the latter approach was employed in order to provide a methodical framework for the derivation of the more general result in the following subsection 2.3. The tensorial approach also provides a unifying view of the different methods of Courant's and Flanders' books.

2.3. *A Region Deformable to Star-Shape External to a Sphere*

The result is now expanded to cover a more general case of a region E deformable to star-shape external to a sphere S. Since E is deformable to D of the previous subsection 2.2, any point Q of E here with coordinates $(\bar{\bar{x}}^1(Q), \bar{\bar{x}}^2(Q), \bar{\bar{x}}^3(Q))$ corresponds to a point P of D of the last subsection with coordinates $(x^1(P) = \bar{\bar{x}}^1(Q), x^2(P) = \bar{\bar{x}}^2(Q), x^3(P) = \bar{\bar{x}}^3(Q))$. Hence, we can form a coordinate system $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$, based on $(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)$, so that every point in E with coordinates $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ can be joined by a line of constant $(\bar{x}^1, \bar{x}^2)$ to its projection point $(\bar{x}^1, \bar{x}^2, R)$, R being a constant, on the internal surface S_1. It is noted that the coordinates $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ are spherical coordinates with respect to the coordinates $(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)$ but are not spherical with respect to the coordinates (x^1, x^2, x^3).

We note that the twice covariant tensor $\mathbf{c}$ is of zero-period and differentiable in any coordinate system. Let $(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)$ serve as the cartesian coordinates; then $\bar{\bar{c}}_{ij}$ is of zero-period and differentiable in a star-shaped region external to the sphere of radius r_0 in this coordinates. Using exactly the same argument as in the previous subsection 2.2, we obtain the solution to the internal Neumann problem for this sphere and have two functions $\bar{\lambda}_1$ and $\bar{\lambda}_2$ on the surface of the sphere such that

$$\frac{\partial \bar{\lambda}_2(\bar{x}^1, \bar{x}^2)}{\partial \bar{x}^1} - \frac{\partial \bar{\lambda}_1(\bar{x}^1, \bar{x}^2)}{\partial \bar{x}^2} = \bar{c}_{12}(\bar{x}^1, \bar{x}^2, r_0).$$

Therefore we have constructed a solution to Eq. (2.11) for this general case of a region deformable to star-shape external to a sphere. (The reader who thinks in terms of vectors arrives at the same results by first proving that $\mathbf{a}[\partial(x^1, x^2, x^3)/\partial(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)]$ is solenoidal and differentiable with $(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)$ serving as cartesian coordinates, and then obtaining a solution to

$$\frac{\partial \bar{\bar{w}}_i}{\partial \bar{\bar{x}}^j} - \frac{\partial \bar{\bar{w}}_j}{\partial \bar{\bar{x}}^i} = (\det(\mathbf{g}))^{1/2}\, \bar{\bar{v}}^k\, \delta^{123}_{ijk}$$

on the surface of the sphere in the $(\bar{\bar{x}}^1, \bar{\bar{x}}^2, \bar{\bar{x}}^3)$ coordinates. He then transforms the results into the required form.)

Hence, for this general case, we still can use the same method to derive all equations up to Eq. (2.11) with only trivial changes to allow for the general (non-spherical) nature of the coordinates $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$. The proof for the existence of a solution to Eq. (2.11) on the internal boundary of the region E has just been given above. Hence the whole method of proof also holds for any solenoidal, differentiable vector field in a region deformable to a star-shaped one external to a sphere. The calculations in the proof from Eqs. (2.12) to (2.14) are slightly modified to take into a account the general (non-spherical) nature of the coordinates $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$, and Eqs. (2.10) do not reduce to the simple form of Eqs. (2.13) but rather take the full tensorial transformation to give

$$\begin{aligned} b_k &= \frac{\partial \bar{x}^1}{\partial x^k} \int_R^{\bar{x}^3} \frac{\partial x^i(\bar{x}^1, \bar{x}^2, \zeta)}{\partial \bar{x}^3} \frac{\partial x^j(\bar{x}^1, \bar{x}^2, \zeta)}{\partial \bar{x}^1} c_{ij}(\bar{x}^1, \bar{x}^2, \zeta)\, d\zeta \\ &\quad - \frac{\partial \bar{x}^2}{\partial x^k} \int_R^{\bar{x}^3} \frac{\partial x^i(\bar{x}^1, \bar{x}^2, \zeta)}{\partial \bar{x}^2} \frac{\partial x^j(\bar{x}^1, \bar{x}^2, \zeta)}{\partial \bar{x}^3} c_{ij}(\bar{x}^1, \bar{x}^2, \zeta)\, d\zeta \\ &\quad + \frac{\partial \bar{x}^1}{\partial x^k} \bar{\lambda}_1(\bar{x}^1, \bar{x}^2) + \frac{\partial \bar{x}^2}{\partial x^k} \bar{\lambda}_2(\bar{x}^1, \bar{x}^2), \qquad \text{where} \quad k = 1, 2, 3. \end{aligned} \tag{2.15}$$

It is worth noting that the (non-spherical) internal boundary of the region E is a smooth surface with finite curvature in the cartesian coordinates (x^1, x^2, x^3), therefore it possesses a solution to the internal Neumann boundary value problem (Kellogg [17, p. 314]) involving $\mathbf{a}(\mathbf{x})$. This, in conjunction with Lichtenstein's or Stevenson's method, gives an alternative approach to establish the existence of the solution $(\bar{\lambda}_1, \bar{\lambda}_2)$ to Eq. (2.11) on the non-spherical, internal surface of the region E. However, this method is not as elementary as the one adopted.

3. Solution in a Region Deformable to a Star-Shaped Region with a Finite Number of Non-intersecting Spherical Cuts

3.1. *A Star-Shaped Region with a Conical Cut*

Let H_1 be the truncated semi-infinite cone defined by

$$r_1 < [(x^1)^2 + (x^2)^2 + (x^3)^2]^{1/2} \qquad \text{and} \qquad \frac{x^3}{[(x^1)^2 + (x^2)^2 + (x^3)^2]^{1/2}} > \cos\alpha.$$

Let H_2 be the truncated semi-infinite cone defined by

$$r_2 \leqslant [(x^1)^2+(x^2)^2+(x^3)^2]^{1/2} \quad \text{and} \quad \frac{x^3}{[(x^1)^2+(x^2)^2+(x^3)^2]^{1/2}} \geqslant \cos\alpha.$$

Let G be a doubly truncated cone defined by

$$r_1 < [(x^1)^2+(x^2)^2+(x^3)^2]^{1/2} < r_2 \quad \text{and} \quad \frac{x^3}{[(x^1)^2+(x^2)^2+(x^3)^2]^{1/2}} > \cos\alpha.$$

We note that $G = H_1 - H_2$. Let D be a star-shaped region centered on the origin O and let us assume that D contains the sphere bounded by the spherical surface S_2 of radius r_2 and is centered on O. Therefore G lies wholly in D. We define $F = D - G$ to be the difference between the star-shaped region D and the doubly truncated cone G. The situation is illustrated in Fig. 3.

Let $\mathbf{a}(\mathbf{x})$ be a given vector field, which is defined, differentiable and is solenoidal in F. We will prove that a continuously differentiable vector field $\mathbf{b}(\mathbf{x})$ can be constructed in F such that its curl is equal to $\mathbf{a}(\mathbf{x})$.

By the proof in the preceeding section, a vector function $\mathbf{b}(\mathbf{x})$ such that

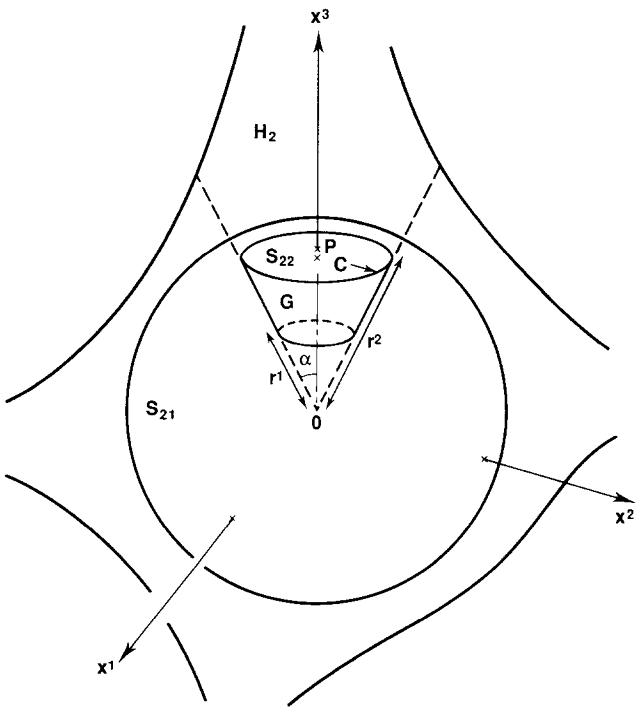

FIG. 3. A star-shaped region D with a conical cut G.

its curl is equal to $\mathbf{a}(\mathbf{x})$ can be constructed for the region $F-H_1$. Our task here is to prove that the function $\mathbf{b}(\mathbf{x})$ so constructed can be extended into the semi-infinite cone H_2.

Since the flux of $\mathbf{a}(\mathbf{x})$ over the spherical surface S_2 is zero, we can construct a vector function $(\bar{\lambda}_1, \bar{\lambda}_2)$ such that we obtain

$$\frac{\partial\bar{\lambda}_2(\bar{x}^1, \bar{x}^2)}{\partial\bar{x}^1} - \frac{\partial\bar{\lambda}_1(\bar{x}^1, \bar{x}^2)}{\partial\bar{x}^2} = \bar{c}_{12}(\bar{x}^1, \bar{x}^2, r_2)$$

over the whole surface S_2. The proof for this has been given in the preceeding section, with the choice that $\bar{x}^3$ is along the radial direction of the (x^1, x^2, x^3) coordinates. Therefore, on the part S_{21} outside the cone H_1 of the spherical surface S_2, we end up with the function $(\bar{\mu}_1, \bar{\mu}_2)$ defined by

$$\bar{\mu}_1(\bar{x}^1, \bar{x}^2) \equiv \bar{b}_1(\bar{x}^1, \bar{x}^2, r_2) - \bar{\lambda}_1(\bar{x}^1, \bar{x}^2)$$

and (3.1)

$$\bar{\mu}_2(\bar{x}^1, \bar{x}^2) \equiv \bar{b}_2(\bar{x}^1, \bar{x}^2, r_2) - \bar{\lambda}_2(\bar{x}^1, \bar{x}^2),$$

which satisfies

$$\frac{\partial\bar{\mu}_2(\bar{x}^1, \bar{x}^2)}{\partial\bar{x}^1} - \frac{\partial\bar{\mu}_1(\bar{x}^1, \bar{x}^2)}{\partial\bar{x}^2} = 0. \tag{3.2}$$

If this vector function $(\bar{\mu}_1, \bar{\mu}_2)$ can be extended such that it has continuous derivatives of first order over the whole surface S_2, as is proved later, then the function $(\bar{\lambda}_1 + \bar{\mu}_1, \bar{\lambda}_2 + \bar{\mu}_2)$, which is defined and continuous over the whole surface S_2, can be considered to be the required extension of $\mathbf{b}(\mathbf{x})$ over the whole surface S_2. Hence the function $\mathbf{d}(\mathbf{x})$ defined by

$$\begin{aligned} d_1 = x^3 \int_{r_2/r}^{1} tc_{31}(t\mathbf{x})\,dt - x^2 \int_{r_2/r}^{1} tc_{12}(t\mathbf{x})\,dt \\ + \frac{\partial\bar{x}^1}{\partial x^1}[\bar{\lambda}_1(\bar{x}^1, \bar{x}^2) + \bar{\mu}_1(\bar{x}^1, \bar{x}^2)] \\ + \frac{\partial\bar{x}^2}{\partial x^1}[\bar{\lambda}_2(\bar{x}^1, \bar{x}^2) + \bar{\mu}_2(\bar{x}^1, \bar{x}^2)], \end{aligned} \tag{3.3a}$$

$$\begin{aligned} d_2 = x^1 \int_{r_2/r}^{1} tc_{12}(t\mathbf{x})\,dt - x^3 \int_{r_2/r}^{1} tc_{23}(t\mathbf{x})\,dt \\ + \frac{\partial\bar{x}^1}{\partial x^2}[\bar{\lambda}_1(\bar{x}^1, \bar{x}^2) + \bar{\mu}_1(\bar{x}^1, \bar{x}^2)] \\ + \frac{\partial\bar{x}^2}{\partial x^2}[\bar{\lambda}_2(\bar{x}^1, \bar{x}^2) + \bar{\mu}_2(\bar{x}^1, \bar{x}^2)], \end{aligned} \tag{3.3b}$$

$$d_3 = x^2 \int_{r_2/r}^{1} tc_{23}(t\mathbf{x})\,dt - x^1 \int_{r_2/r}^{1} tc_{31}(t\mathbf{x})\,dt$$
$$+\frac{\partial \bar{x}^1}{\partial x^3}\left[\bar{\lambda}_1(\bar{x}^1, \bar{x}^2) + \bar{\mu}_1(\bar{x}^1, \bar{x}^2)\right]$$
$$+\frac{\partial \bar{x}^2}{\partial x^3}\left[\bar{\lambda}_2(\bar{x}^1, \bar{x}^2) + \bar{\mu}_2(\bar{x}^1, \bar{x}^2)\right], \tag{3.3c}$$

is identical to the function $\mathbf{b}(\mathbf{x})$ in the region $F - H_1$. This function $\mathbf{d}(\mathbf{x})$ is also continuous and differentiable in the whole part of the region F outside the surface S_2 and is thus the required extension of the function $\mathbf{b}(\mathbf{x})$ into the whole region F.

We now proceed to prove that the vector function $(\bar{\mu}_1, \bar{\mu}_2)$ can be extended such that it has continuous derivatives of first order over the whole surface S_2. The circle C on S_2 defined by $x^3/[(x^1)^2 + (x^2)^2 + (x^3)^2]^{1/2} = \cos\alpha$ divides this surface into two spherical caps S_{21} and S_{22}, which lie outside and inside H_1, respectively. Both of them are simply connected. On the surface S_{21}, we can construct a potential function $\bar{\psi}(\bar{x}^1, \bar{x}^2)$ from the two-dimensional, irrotational vector $(\bar{\mu}_1, \bar{\mu}_2)$ as

$$\bar{\psi}(\bar{x}^1, \bar{x}^2) = \int (\bar{\mu}_1\,d\bar{x}^1 + \bar{\mu}_2\,d\bar{x}^2). \tag{3.4}$$

This function is defined and has continuous derivatives of second order on S_{21}.

On the spherical cap S_{22}, with its pole P at position ($x^1 = 0$, $x^2 = 0$, $x^3 = r_2$), we define a function $\bar{\sigma}(\bar{x}^1, \bar{x}^2)$ by drawing great circles through P, which intersect the circle C at a right angle. The value of the function $\bar{\sigma}$ at a point on S_{22} at a polar angle β from P is given by

$$\bar{\sigma}(\bar{x}^1(\beta, \theta), \bar{x}^2(\beta, \theta)) = [L(\theta)]\,\beta^5 + [M(\theta)]\,\beta^4 + [N(\theta)]\,\beta^3, \tag{3.5}$$

where θ is the longitudinal angle of the great circular arc originating from the pole P and the coefficients $L(\theta)$, $M(\theta)$, and $N(\theta)$ are chosen such that the function $\bar{\sigma}(\bar{x}^1(\beta, \theta), \bar{x}^2(\beta, \theta))$ and its first and second derivatives with respect to β are equal to those respective values of $\bar{\psi}$ for $\beta = \alpha$. Its second derivatives on the spherical cap S_{22} are given by

$$\frac{\partial^2 \bar{\sigma}}{\partial \beta^2} = 20L\beta^3 + 12M\beta^2 + 6N\beta,$$
$$\frac{\partial^2 \bar{\sigma}}{\partial \theta\,\partial \beta} = \frac{\partial^2 \bar{\sigma}}{\partial \beta\,\partial \theta} = 5\frac{\partial L}{\partial \theta}\beta^4 + 4\frac{\partial M}{\partial \theta}\beta^3 + 3\frac{\partial N}{\partial \theta}\beta^2,$$
$$\frac{\partial^2 \bar{\sigma}}{\partial \theta^2} = \frac{\partial^2 L}{\partial \theta^2}\beta^5 + \frac{\partial^2 M}{\partial \theta^2}\beta^4 + \frac{\partial^2 N}{\partial \theta^2}\beta^3,$$

which are defined and continuous on all of S_{22}, including at the pole P.

On the spherical cap S_{21}, we define $\bar{\sigma} = \bar{\psi}$. The function $\bar{\sigma}$ is thus defined on the whole of S_2 and is twice differentiable on S_2, including at the pole P. Therefore, we have an extension of the function $\bar{\psi}$ onto the whole of S_2. Hence the vector function $(\bar{\mu}_1, \bar{\mu}_2)$ has been extended into a vector function $(\bar{v}_1, \bar{v}_2)$ given by

$$\bar{v}_1(\bar{x}^1, \bar{x}^2) = \frac{\partial \bar{\sigma}}{\partial \bar{x}^1} \quad \text{and} \quad \bar{v}_2(\bar{x}^1, \bar{x}^2) = \frac{\partial \bar{\sigma}}{\partial \bar{x}^2}, \tag{3.6}$$

which is defined and differentiable over the whole surface S_2 and is identical to $(\bar{\mu}_1, \bar{\mu}_2)$ on S_{21}.

Thus we have proved that Eq. (2.1) has a solution in the region F.

If the whole of S cannot be drawn inside D, we only need to modify S_{21} into a smooth surface which lies wholly inside D and which joins smoothly into S_{22}. We then construct the functions $\bar{\lambda}_1, \bar{\lambda}_2$ on the closed, smooth surface formed by the union of S_{22} and the above smooth surface, and follow the same argument of the previous case. The final result remains the same.

3.2. *A Star-Shaped Region with Spherical Cuts*

Let the region $M = D - H$ be the difference between a star shaped region D centered on the origin O of the coordinate system (x^1, x^2, x^3) and the interior H of a spherical surface T centered on W, with O outside T. Let h be the distance from the origin O to the center W of the sphere T and let R be the radius of this sphere. We put

$$r_1 = h - R, \qquad r_2 = h + R, \qquad \text{and} \qquad \alpha = \arcsin\left(\frac{R}{h}\right)$$

and we suppose, for the time being, that we can enclose the sphere T here by the doubly truncated cone G defined as in the last subsection. We can apply the same technique as used previously to derive the result for the region M. The proof proceeds as in the last subsection by noting that the vector function $\mathbf{d}(\mathbf{x})$ can be defined into the region interior to G but exterior to T by allowing the magnitude $|\mathbf{x}|$ of $\mathbf{x}$ of formulae (3.3) to have values less than r_2.

If we cannot draw a doubly truncated cone G around T then the following procedure is used: Enclose T by another concentric spherical surface U of slightly larger radius than T (This is possible as T is disjointed from the external boundary of D.) Apply the deformation described immediately after Definition 5, so that the part of D outside U is unaltered, but the region between T and U is deformed into the region between V and U, where V is another concentric spherical surface of radius one-half that of U. Finally enclose V in a doubly truncated cone G, which lies wholly inside U, therefore also wholly in D, and apply the reasoning in the previous paragraph (with some trivial change of coordinates) to the deformed region.

The result is: For any solenoidal, differentiable vector $\mathbf{a}(\mathbf{x})$ in such a region M, there is a vector function $\mathbf{b}(\mathbf{x})$, the curl of which is equal to $\mathbf{a}(\mathbf{x})$. The result here is the same as that of the last section. However, this section has shown that the result need not be the consequence of the center of the spherical internal boundary being the center of the coordinate system (x^1, x^2, x^3). The implication is that we can use the technique given here for a region that is the difference between a star-shaped region D and a finite number of non-intersecting spheres, all of which lie wholly in D.

We can then repeat the whole argument, with some trivial change of coordinates, for any region deformable to a star-shaped one with spherical cuts, each of which is external to every other and is wholly within the star-shaped region.

As this proof constructs the vector function $\mathbf{b}(\mathbf{x})$ from the origin outwards, it can even be extended to cover some cases where a region is the difference between a star-shaped region D and an infinite but countable number of non-intersecting spheres, all of which lie wholly in D. The method is to insert one internal boundary after another, in order of increasing $\bar{x}^3$, into the region under consideration.

4. Comparison with Monge's Potentials

The solution to Eq. (2.1) has also been obtained alternatively by expressing the vector $\mathbf{a}(\mathbf{x})$ as the vector product of two gradients, i.e.,

$$\mathbf{a} = \nabla\lambda_1 \times \nabla\lambda_2. \tag{4.1}$$

If such an expression is successfull then the required vector $\mathbf{b}(\mathbf{x})$ is given simply by

$$\mathbf{b} = \lambda_1 \nabla\lambda_2. \tag{4.2}$$

This method was used by Clebsch [13, 14] in a work now known as Clebsch' transformation (see also Lamb's book [15]). It has been adopted by Phillips [18], and has also been adopted in a recent book by Aris [19]. Aris called the quantities λ_1 and λ_2 "Monge's potential" (after Gaspard Monge, 1746–1810). The method is summarised as follows:

The physical meaning of this method is that $\mathbf{a}(\mathbf{x})$ forms a system of never ending tubes in the three-dimensional space (x^1, x^2, x^3). A system of surfaces $\bar{x}^1(\mathbf{x}) = constant$ and $\bar{x}^2(\mathbf{x}) = constant$ is constructed along these $\mathbf{a}(\mathbf{x})$

tubes so that $\mathbf{a}$ is along the intersections of these two families of surfaces, i.e.,

$$\mathbf{a} = \phi \nabla \bar{x}^1 \times \nabla \bar{x}^2, \tag{4.3}$$

where ϕ is a scalar variable. Let $\bar{x}^3(\mathbf{x})$ be a third function of (x^1, x^2, x^3) which together with $\bar{x}^1$ and $\bar{x}^2$ forms a system of curvilinear coordinates in (x^1, x^2, x^3). The condition that $\mathbf{a}(\mathbf{x})$ must be differentiable and divergence-free then gives $\partial\bar{\phi}/\partial\bar{x}^3 = 0$; i.e., $\bar{\phi} = \bar{\phi}(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ is a function of only $\bar{x}^1$ and $\bar{x}^2$. We then proceed to find a two-dimensional function $(\bar{\lambda}_1(\bar{x}^1, \bar{x}^2), \bar{\lambda}_2(\bar{x}^1, \bar{x}^2))$ such that

$$\frac{\partial \bar{\lambda}_2(\bar{x}^1, \bar{x}^2)}{\partial \bar{x}^1} - \frac{\partial \bar{\lambda}_1(\bar{x}^1, \bar{x}^2)}{\partial \bar{x}^2} = \bar{\phi}(\bar{x}^1, \bar{x}^2). \tag{4.4}$$

Substituting the value of ϕ from Eq. (4.4) into Eq. (4.3) then gives

$$a_i = \phi \delta_{ijk}^{123} \frac{\partial(\bar{x}^1, \bar{x}^2)}{\partial(x^j, x^k)} = \delta_{ijk}^{123} \frac{\partial(\bar{x}^1, \bar{x}^2)}{\partial(x^j, x^k)} \frac{\partial(\lambda_1, \lambda_2)}{\partial(\bar{x}^1, \bar{x}^2)} = \delta_{ijk}^{123} \frac{\partial(\lambda_1, \lambda_2)}{\partial(x^j, x^k)}$$

or

$$\mathbf{a} = \nabla \lambda_1 \times \nabla \lambda_2,$$

as required by (4.1).

The derivation of this method up to Eq. (4.3) follows the method of Phillips [18] and Aris [19]. (Clebsch' work [13, 14] used a more complicated method to prove that ϕ is a function of only $\bar{x}^1$ and $\bar{x}^2$.) The substitution of the solution to Eq. (4.4) into Eq. (4.3) is a re-expression of Clebsch' [13, 14] argument (as given by Lamb [15]) in modern notation. (The proofs in Phillips and Aris' books gave some explicit simple solutions to Eq. (4.4) for only some non-general cases.) In the present author's opinion, such a combination of arguments from the above two sources appears to offer simplicity while retaining the generality of the method.

This method appears superficially to be simpler than those given in the previous two sections. However, careful examination reveals that the method requires first that there exists a system of integration surfaces $\bar{x}^1$ and $\bar{x}^2$ such that every integration line of $\mathbf{a}(\mathbf{x})$ has a unique value of $(\bar{x}^1, \bar{x}^2)$ and second that Eq. (4.4) has a solution. The proof for the former is not a trivial matter. This important point appears to have been overlooked in the works using this method [14, 15, 18, 19]. It is also noted that the use of the solution to Eq. (4.4) in this method is similar to the use of the solution to Eq. (2.11) in the methods in Sections 2 and 3.

5. Applications

It might be expected that a line integration method exists for the related problem of the Stokes–Helmholtz decomposition theorem. However, we can show here that such an expectation is unjustified. Indeed, for an arbitrarily given function $\psi(\mathbf{x})$, we can form a vector function

$$\mathbf{a}(\mathbf{x}) = \left(0, 0, \int_0^{x^3} \psi(x^1, x^2, \zeta)\, d\zeta\right),$$

provided that the region under consideration is x^3-convex. If there were a line integral method, in a manner similar to that of section 2, for the determination of the potential $\phi(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ such that $\mathbf{a}(\mathbf{x})$ has its Stokes–Helmholtz decomposition

$$\mathbf{a} = \nabla\phi + \nabla \times \mathbf{b}$$

then we would have found a solution to the Poisson Eq. $\nabla^2\phi = \psi$ by a line integral method. This is absurd as $\phi(\mathbf{x})$ depends on all values of $\psi(\mathbf{x})$.

Thus we must settle for a more restricted result: In a simply connected region F deformable to a star-shaped one with a finite number of spherical cuts, any arbitrarily given continuous and differentiable vector field $\mathbf{a}(\mathbf{x})$ can be expressed as the sum of the gradient of a scalar function $\phi(\mathbf{x})$ and the curl of another vector function $\mathbf{b}(\mathbf{x})$, i.e.,

$$\mathbf{a} = \nabla\phi + \nabla \times \mathbf{b} \tag{5.1}$$

provided that the volume integral

$$\phi(\mathbf{x}) \equiv \frac{-1}{4\pi} \int_H (\nabla \cdot \mathbf{a}) \left(\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{|\mathbf{y}|}\right) d\tau(\mathbf{y}) \tag{5.2}$$

over the intersection H of F and the interior of a sphere S of radius r_3 converges as r_3 tends to infinity.

The theorem is proved by showing that $\phi(\mathbf{x})$ is a solution to the equation $\nabla^2\phi = \nabla \cdot \mathbf{a}$ and then by applying the result of Section 3 to the solenoidal, differentiable vector function $(\mathbf{a} - \nabla\phi)$. The scalar function $\phi(\mathbf{x})$ may need to be added with a finite number of terms of the form $1/|\mathbf{x}-\mathbf{h}|$, where each $\mathbf{h}$ is a constant position vector of some point inside each internal boundary, to make the function $(\mathbf{a} - \nabla\phi)$ solenoidal.

Since

$$-4\pi\phi(\mathbf{x}) = -\int_H \mathbf{a} \cdot \left[\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{|\mathbf{y}|}\right)\right] d\tau(\mathbf{y}) + \int_{S \equiv \partial H} \left(\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{|\mathbf{y}|}\right) \mathbf{a} \cdot \mathbf{n}\, d\sigma(\mathbf{y})$$

it is easily seen that the integral $\phi(\mathbf{x})$ converges when $\mathbf{a}(\mathbf{x})$ vanishes at infinity to the order of $O(|\mathbf{x}|^{-\varepsilon})$, where ε is a positive number. The result here is consistent with a result previously given by Gurtin [20] for a more general region which can be multiply connected.

The condition on the vanishment of $\mathbf{a}(\mathbf{x})$ at infinity can be partially relaxed. This is done by imposing some requirement on the behavior of the derivatives of the components of $\mathbf{a}$ at infinity. We use the result by Blumenthal [21] that the vector function $\mathbf{c}(\mathbf{x})$ given by

$$\mathbf{c}(\mathbf{x}) \equiv \frac{1}{4\pi} \int_F (\nabla \cdot \mathbf{a}) \nabla \left(\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{|\mathbf{y}|} \right) d\tau(\mathbf{y})$$

is defined when $|\mathbf{a}(\mathbf{x})| = O(|\mathbf{x}|^{1/2-\varepsilon})$ and $\partial a_i/\partial x^j = O(|\mathbf{x}|^{1/2-\varepsilon})$, $i, j = 1, 2, 3$, and satisfies $\nabla \cdot (\mathbf{a} - \mathbf{c}) = 0$ and $\nabla \times \mathbf{c} = 0$. Incorporating such a result, we can state that any arbitrarily given function $\mathbf{a}(\mathbf{x})$ which is continuous and differentiable in a simply connected region F, which is deformable to a star-shaped one with a finite number of spherical cuts, can be expressed in the form (5.1) provided that the components a_i, $i = 1, 2, 3$, of $\mathbf{a}(\mathbf{x})$ and their derivatives $\partial a_i/\partial x^j$, $i, j = 1, 2, 3$, are of order $O(|\mathbf{x}|^{1/2-\varepsilon})$ when $|\mathbf{x}|$ tends to infinity. These results give another kind of sufficient condition for the application of the Helmholtz decomposition theorem besides the well-known one, as given in Phillips' book [18], which requires that $\mathbf{a}(\mathbf{x})$ vanish to the order of $O(|\mathbf{x}|^{-2-\varepsilon})$ as $|\mathbf{x}|$ tends to infinity.

A practical application of the results from this paper is the extension of the proof for the gauge transformation of dynamic electricity, such as that given in Lass' book (pp. 175–177), to cover any arbitrary simply connected region of the three-dimensional space (x, y, z) with a finite number of interior surfaces. The existing proof given in that book was written on the basis of a result (p. 117) for only simply connected regions of the three-dimensional space (x, y, z) which are convex in the x and z directions.

A problem related to the results derived in Section 2 and 3 is the application of the same line integration technique used in those sections to the proof of some decomposition theorems for a second order tensor in a region G deformable to star-shape: We use bold lower case roman letters to denote vector functions, bold upper case romans letters to denote tensors of second order and higher, and the contraction $\mathbf{C} = \mathbf{A} \cdots \mathbf{B}$ to give the components of $\mathbf{C}$ as $C_{inp} = A_{ijkl} B_{lkjnp}$. The tensors $\mathbf{E}$ and $\mathbf{I}$ are third and second order tensors with components $E_{ijk} = \delta^{123}_{ijk}$ and $I_{ij} = \delta^j_i$, where δ^j_i is a Kronecker δ function. $\mathbf{S}$ denotes a given second-order tensor. The notation $\mathbf{S} \in C^2(G)$ means that the tensor $\mathbf{S}$ has continuous second derivatives in G. We then have the following four results:

(a) If $\nabla \times \mathbf{S} \times \nabla = 0$ and $\mathbf{S}^T = \mathbf{S} \in C^2(G)$ then $\nabla \times \mathbf{S} = \mathbf{t}\nabla$. Therefore $\nabla \cdot \mathbf{t} = \mathbf{I} \cdot\cdot (\mathbf{t}\nabla) = \mathbf{I} \cdot\cdot (\nabla \times \mathbf{S}) = -\mathbf{I} \cdot\cdot (\mathbf{E} \cdot\cdot \nabla\mathbf{S}) = -\mathbf{E} \cdots \nabla\mathbf{S} = -\frac{1}{2}\mathbf{E} \cdots \nabla\mathbf{S} - \frac{1}{2}\mathbf{E} \cdots \nabla\mathbf{S}^T = \frac{1}{2}(-\mathbf{E} + \mathbf{E}) \cdots \nabla\mathbf{S} = 0$. Hence $\nabla \times \mathbf{S} = \nabla \times \mathbf{v}\nabla$. Finally, $\mathbf{S} = \mathbf{v}\nabla + \nabla\mathbf{w} = \nabla\mathbf{u} + \mathbf{u}\nabla$, where $\mathbf{u} \in C^3(G)$.

(b) If $\nabla\cdot\mathbf{S}\cdot\nabla=0$ and $\mathbf{S}^T=\mathbf{S}\in C^2(G)$ then $\nabla\cdot\mathbf{S}=-\mathbf{u}\times\nabla=\mathbf{u}\nabla\cdot\cdot\mathbf{E}=-\nabla\cdot\mathbf{E}\cdot\mathbf{u}$, so that $\mathbf{S}\cdot\nabla=\nabla\cdot\mathbf{S}^T=\nabla\cdot\mathbf{S}=\mathbf{u}\cdot\mathbf{E}\cdot\nabla$. Thus $\nabla\cdot(\mathbf{S}+\mathbf{E}\cdot\mathbf{u})=(\mathbf{S}-\mathbf{u}\cdot\mathbf{E})\cdot\nabla=0$, so that $\mathbf{S}=-\mathbf{E}\cdot\mathbf{u}+\nabla\times\mathbf{W}=\mathbf{u}\cdot\mathbf{E}+\mathbf{T}\times\nabla$. Finally, $\mathbf{S}=\frac{1}{2}(\nabla\times\mathbf{W}+\mathbf{T}\times\nabla)=\nabla\times\mathbf{A}-\mathbf{A}^T\times\nabla$, where $\mathbf{A}\in C^3(G)$.

(c) If $\mathbf{S}$ is solenoidal and $\mathbf{S}^T=\mathbf{S}\in C^1(G)$ then $\mathbf{S}=\nabla\times\mathbf{U}$, so that $0=\mathbf{S}\cdot\nabla=\nabla\times(\mathbf{U}\cdot\nabla)$. Then $\mathbf{U}\cdot\nabla=(f\mathbf{I})\cdot\nabla$, $\mathbf{U}=f\mathbf{I}+\mathbf{V}\times\nabla$. Hence $\mathbf{S}=\nabla\times\mathbf{V}\times\nabla+\nabla\times(f\mathbf{I})$ and $\mathbf{S}^T=\nabla\times\mathbf{V}^T\times\nabla+(f\mathbf{I})^T\times\nabla=\nabla\times\mathbf{V}^T\times\nabla-\nabla\times(f\mathbf{I})$. Finally, $\mathbf{S}=\nabla\times\mathbf{B}\times\nabla$, where $\mathbf{B}\in C^3(G)$.

(d) If $\nabla\times\mathbf{S}=0$ and $\mathbf{S}^T=\mathbf{S}\in C^1(G)$ then $\mathbf{S}=\nabla\mathbf{a}$, so that $0=\mathbf{S}\times\nabla=\nabla(\mathbf{a}\times\nabla)$. Thus $\mathbf{a}\times\nabla=2\mathbf{m}=(\mathbf{x}\cdot\mathbf{E}\cdot\mathbf{m})\times\nabla$, where $\mathbf{m}$ is a constant vector. Hence $\mathbf{a}=(\mathbf{x}\cdot\mathbf{E}\cdot\mathbf{m})+f\nabla$. Finally, $\mathbf{S}=\nabla f\nabla+\mathbf{E}\cdot\mathbf{m}=\nabla f\nabla$, where $f\in C^3(G)$ as $\mathbf{S}^T=\nabla f\nabla-\mathbf{E}\cdot\mathbf{m}$.

Results (a) and (b) are the duals to each other from the view of calculus of differential forms. So are (c) and (d). Their interesting feature is the conciseness of their derivations. Result (a) is readily recognized as the Saint-Venant's compatibility of a strain tensor in elasticity. An early proof to the same effect was given by Cesaro [22] (see Sokolnikoff [23]); his proof is more general but is also much more complicated than this one and is usually left out of textbooks on that subject. Result (c) is the justification for Maxwell's and Morera's stress functions.

6. Conclusions

The solution to the problem of expressing a solenoidal, differentiable vector as the curl of another vector has been given for regions deformable to star-shape with a finite number of spherical cuts. The method of solution here is shown to be more rigorous than the method using Monge's potentials.

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